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THE KRULL ORDINAL AND LENGTH OF A NOETHERIAN MODULE

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0. Introduction

Let M be a non-zero Noetherian module over a commutative ring R with identity. Denote by $\mathcal{L}(M)$ the Krull ordinal of the set $S(M)$ of all submodules of M (see below). We find an expression for $\mathcal{L}(M)$ in terms of certain isolated components of 0_M. As a by-product we obtain an improvement of [1, Theorem 2.12], relating the Krull dimension $\kappa(M)$ of M to $o(M)$, the supremum of the ordinal types of chains in $S(M)\ 0_M$. Such chains are always countable, by [1, §1]. However, it has been shown in [2] (and also by T. Gulliksen) that for any ordinal d there exists a ring of Krull dimension d. When $\kappa(M) < \Omega$ (where Ω denotes the least uncountable ordinal), we find that $\mathcal{L}(M)$ equals $o(M)$ and that this is in fact the ordinal type of a longest chain in $M\ 0_M$. However, in the case $\kappa(M) \geqslant \Omega$, whilst $o(M)$ is the constant Ω, $\mathcal{L}(M)$ remains sensitive to change in $\kappa(M)$.

1. Preliminaries

The reader is referred to [1, §2.6] for the definition of the *Krull ordinal* $\kappa(T)$ of a partially ordered set T. In the cases with which we are concerned, T is either Supp M or the set $S(M)$ of all submodules of M, and in each case the ordering is reverse inclusion $\supseteq$. Thus T satisfies the Noetherian condition and so $\kappa(T)$ does exist. We observe that

$$\kappa(T) = \sup\{f(t) \colon t \in T\},$$

where f is the least ordinal-valued function on T for which $f(A) > f(B)$ whenever $A \subset B$. If U and V are subsets of T such that $u \supseteq v$ whenever $u \in U$ and $v \in V$, then

$$\kappa(T) \geqslant \kappa(U) + \kappa(V).$$

Thus, when $\mathcal{L}$ is defined by $\mathcal{L}(M) = \kappa(S(M))$, we have

$$\mathcal{L}(M) \geqslant \mathcal{L}(M/N) + \mathcal{L}(N)$$

for each submodule N of M. The *Krull dimension* (termed "classical" in [2]) of the module M is $\kappa(\operatorname{Supp} M)$, and we write this as $\kappa(M)$.

We intend to define an ordinal-valued function $\mathcal{P}$ on the set of factor modules of M. Let I be any proper submodule of M, and let $d_1 > \ldots > d_h$ be the distinct values taken by $\kappa(R/P)$ when $P \in \operatorname{Ass}(M/I)$. For $i = 1, \ldots, h$, denote by S_i the complement of the union of all primes $P \in \operatorname{Ass}(M/I)$ for which $\kappa(R/P) = d_i$, and let I_i denote the intersection of all primary components Q, in a normal primary decomposition of I, of which the associated prime P (= Rad Q) has $\kappa(R/P) \geqslant d_i$. Each I_i is independent of the particular primary decomposition of I, and putting $I_0 = M$ we have

$$I_0 \supset I_1 \supset \ldots \supset I_{h-1} \supset I_h = I\,.$$

For any multiplicatively closed subset S of R, we shall denote the length of the R_S-module M_S by $\ell_S(M)$. For $i = 1, \ldots, h$, we write

$$m_i = \ell_{S_i}(I_{i-1}/I)\,.$$

Note that, given a normal primary decomposition of I in M, the primary submodules belonging to primes P such that $\kappa(R/P) \leqslant d_i$ provide, when intersected with I_{i-1}, a normal primary decomposition of I in I_{i-1}. Thus $\kappa(I_{i-1}/I) = d_i$ and so $1 \leqslant m_i < \omega$. Further it will be useful to note that m_i is the length of any saturated chain in I_{i-1}/I whose terms are intersections of primary submodules belonging to primes P in $\operatorname{Ass}(I_{i-1}/I)$ such that $\kappa(R/P) = d_i$. We define $\mathcal{P}$ by

$$\mathcal{P}(M/M) = 0$$

and, for each proper submodule I,

$$\mathcal{P}(M/I) = \omega^{d_1} m_1 + \omega^{d_2} m_2 + \ldots + \omega^{d_h} m_h\,.$$

Of course, h, S_i, d_i, etc., depend on I, and to indicate this we shall write $h(I)$, $S_i(I)$, $d_i(I)$, etc.

Denote by $\ell(M)$ the maximum (if it exists) of the ordinal types of chains of non-zero R-submodules of M.

2.

Theorem. (i) *For all values of* $\kappa(M)$, $\mathcal{L}(M) = \mathcal{P}(M)$.

(ii) *If* $\kappa(M) < \Omega$, *then* $\ell(M)$ *is defined and* $\ell(M) = o(M) = \mathcal{P}(M)$.

Remarks. (a) Since any chain is countable, $\ell(M)$ is undefined and $o(M) = \Omega$ when $\kappa(M) \geqslant \Omega$ (cf. [1, Remark 2.13]).

(b) It is easy to deduce [1, Theorem 2.12] from (ii). Our proof does not involve the result [1, Corollary 3.7] about ordinals.

(c) There are similar proofs for (i) and (ii). We shall prove (i) and will indicate in square brackets the modifications required to obtain a proof of (ii).

Proof of (i). First we show that $\mathcal{P}(M) \geqslant \mathcal{L}(M)$. It is sufficient to prove that if J and I are R-submodules of M such that $J \subset I$, then $\mathcal{P}(M/I) < \mathcal{P}(M/J)$. [Clearly, this will entail $\mathcal{P}(M) \geqslant o(M)$.] We shall, in fact, assume the hypotheses $J \subseteq I$ and $\mathcal{P}(M/I) \geqslant \mathcal{P}(M/J)$, and will show that these imply $J = I$. Suppose inductively that $I_0 = J_0$ $(= M)$, $I_1 = J_1, \dots, I_n = J_n$, where $0 \leqslant n < h(J)$. We require to prove that I_{n+1} is defined and that $I_{n+1} = J_{n+1}$. From this it will follow that $J_{h(J)} = I_{h(J)}$; but $J_{h(J)} = J \subseteq I \subseteq I_{h(J)}$, and so $J = I$. Our supposition implies that, for $i = 1, \dots, n$,

$$d_i(I) = d_i(J)\,, \qquad m_i(I) = m_i(J)\,,$$

and also that $d_{n+1}(J)$ is defined. Since $\mathcal{P}(M/I) \geqslant \mathcal{P}(M/J)$, it follows that

$$\omega^{d_{n+1}(I)}\, m_{n+1}(I) \geqslant \omega^{d_{n+1}(J)}\, m_{n+1}(J)\,. \tag{1}$$

In particular, the left-hand side of (1) is defined, and so $d_{n+1}(I)$ and hence I_{n+1} are defined.

Let $P \in \operatorname{Ass}(M/I)$ and $\kappa(R/P) = d_{n+1}(I)$. Thus $P \in \operatorname{Ass}(I_n/I)$; we also observe that P is a typical member of $\operatorname{Ass}(I_n/I_{n+1})$. Then

$$P \supseteq I : I_n \supseteq J : J_n\,.$$

But $d_{n+1}(J) \leqslant d_{n+1}(I)$, by (1). Therefore P is minimal over $J : J_n$ and hence $P \in \operatorname{Ass}(J_n/J)$. We now infer that $d_{n+1}(J) = d_{n+1}(I)$, $S_{n+1}(I) \supseteq S_{n+1}(J)$ and $\operatorname{Ass}(I_n/I_{n+1}) \subseteq \operatorname{Ass}(J_n/J_{n+1})$. Furthermore, the P-primary component of J in J_n $(= I_n)$ is contained in the P-primary component of I in I_n, and so $J_{n+1} \subseteq I_{n+1}$. We have

$$\begin{aligned}
m_{n+1}(I) &= \ell_{S_{n+1}(I)}(I_n/I_{n+1}) \\
&\leqslant \ell_{S_{n+1}(I)}(I_n/I_{n+1}) + \ell_{S_{n+1}(I)}(I_{n+1}/J_{n+1}) && (2)\\
&= \ell_{S_{n+1}(I)}(J_n/J_{n+1}) \\
&\leqslant \ell_{S_{n+1}(J)}(J_n/J_{n+1}) && (3)\\
&= m_{n+1}(J)\,.
\end{aligned}$$

However, (1) gives

$$m_{n+1}(I) \geqslant m_{n+1}(J),$$

and hence (2) and (3) are equalities. The equality at (2) implies that the $S_{n+1}(I)$-component of J_{n+1} is I_{n+1}, and the equality at (3) implies that $\mathrm{Ass}(I_n/I_{n+1}) = \mathrm{Ass}(J_n/J_{n+1})$, and so $S_{n+1}(I) = S_{n+1}(J)$. But J_{n+1} is the $S_{n+1}(J)$-component of J_{n+1} in J_n, and so $J_{n+1} = I_{n+1}$.

To show that $\mathcal{P}(M) \leqslant \mathcal{L}(M)$, we may assume inductively that if N is a Noetherian R-module for which $\mathcal{P}(N) < \mathcal{P}(M)$, then $\mathcal{P}(N) \leqslant \mathcal{L}(N)$. [For (ii), our aim is to show that, when $\kappa(M) < \Omega$, there is a chain of non-zero submodules of M of ordinal type $\mathcal{P}(M)$. Taken in conjunction with what has already been proved, this would imply (ii). We assume the result true for N such that $\mathcal{P}(N) < \mathcal{P}(M)$.] Denote $\kappa(M)$ by d. Clearly we may assume $\mathcal{P}(M) > 1$.

Case 1. Suppose that $\mathcal{P}(M) = \omega^d$, where $d = c + 1$ for some c. Then there exist prime ideals P and P_0 such that $P \supset P_0 \supseteq \mathrm{Ann}\, M$ and $\kappa(R/P) = c$. But, for $n \geqslant 0$, $\mathrm{Ann}\, P^n M \subseteq P_0$ and so, by [4, p. 232, Exercise 8], $P^{n+1}M : P^n M = P$. Therefore $\omega^d > \mathcal{P}(P^nM/P^{n+1}M) \geqslant \omega^c$, and hence by our assumption, $\mathcal{L}(P^nM/P^{n+1}M) \geqslant \omega^c$ for $n \geqslant 0$. Therefore $\mathcal{L}(M) \geqslant \omega^c \cdot \omega = \mathcal{P}(M)$. [For (ii), we see that, for each n, there is a chain of type ω^c between P^nM and $P^{n+1}M$. The union of these chains is a chain of type ω^d.]

Case 2. Suppose that $\mathcal{P}(M) = \omega^d$, where d is a limit ordinal. Let e be any ordinal less than d. Then there is a prime ideal P such that $P \supset \mathrm{Ann}\, M$ and $\kappa(R/P) = e$. But $PM : M = P$ and so $\omega^d > \mathcal{P}(M/PM) \geqslant \omega^e$. Therefore $\mathcal{L}(M/PM) \geqslant \omega^e$, and so $\mathcal{L}(M) \geqslant \omega^e$ for all $e < d$. Therefore $\mathcal{L}(M) \geqslant \omega^d = \mathcal{P}(M)$. [For (ii), take a sequence e_n which increases strictly to the limit d, and choose primes P_n such that $P_n \supset \mathrm{Ann}\, M$ and $\kappa(R/P_n) = e_n$. Then $P_1 \ldots P_{n-1}P_nM : P_1 \ldots P_{n-1}M = P_n$ and so, for each n, there exist chains of type ω^{e_n} between $P_1 \ldots P_{n-1}M$ and $P_1 \ldots P_{n-1}P_nM$. The union of these chains has type ω^d.]

Case 3. Suppose that $\mathcal{P}(M) = \omega^d + b$, where $0 < b < \omega^{d+1}$. Choose a prime ideal P such that $P \in \mathrm{Ass}\, M$ and $\kappa(R/P) = d$, and let Q be a maximal P-primary submodule of M. Then $Q \supseteq (0_M)_1$ (where we are taking $I = 0_M$ in the notation introduced before the statement of the theorem). Note that a normal primary decomposition of 0_M (in M) induces a primary decomposition of 0_Q (in Q). If $Q = (0_M)_1$, then it is easy to see that

$$\mathcal{P}(Q) = \sum_{n \geqslant 2} \omega^{d_n(0_M)}\, m_n(0_M);$$

also $\ell_{S_1(0_M)}(M) = 1$ and so $\mathcal{P}(Q) = b$. Now suppose that $Q \supset (0_M)_1$. Then we contend that

$$\ell_{S_1(0_Q)}(Q) = \ell_{S_1(0_M)}(Q). \tag{4}$$

If $P \in \operatorname{Ass} Q$, this is trivial, and otherwise $P \not\supseteq \operatorname{Ann} Q$, in which case Q has no P-primary submodules. From (4) it follows that

$$\ell_{S_1(0_M)}(M) = 1 + \ell_{S_1(0_Q)}(Q),$$

and again we deduce that $\mathcal{P}(Q) = b$. We now have $\mathcal{P}(Q) = b < \mathcal{P}(M)$ and $\mathcal{P}(M/Q) = \omega^d < \mathcal{P}(M)$. Therefore

$$\mathcal{P}(M) = \mathcal{P}(M/Q) + \mathcal{P}(Q) \leqslant \mathcal{L}(M/Q) + \mathcal{L}(Q) \leqslant \mathcal{L}(M).$$

[For (ii), we infer the existence of chains of type ω^d between M and Q and of type b in Q, and hence a chain of type $\mathcal{P}(M)$ in M.]

3. Concluding remarks

It is not in general true that if N is a submodule of M, then $\mathcal{L}(M) = \mathcal{L}(M/N) + \mathcal{L}(N)$.

Dr. J.C. Robson has mentioned to the author that methods similar to those above may be applied over non-commutative rings by the use of quotient categories.

Added in proof. Gulliksen [3] has investigated $\mathcal{L}(M)$ for a Noetherian module M over an arbitrary ring with identity. In particular he showed that if $\kappa(M) < \Omega$ then $\ell(M) = o(M) = \mathcal{L}(M)$. Over a commutative ring he showed that, when $\mathcal{L}(M)$ is written in normal form as a polynominal in ω, the exponents occurring may be expressed in terms of Ass M.

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